Tower of Hanoi

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The eminent Donald Knuth, in "All Questions Answered", confessed that he couldn't prove an optimal solution to the $k = 4$ peg Tower of Hanoi puzzle even after "a solid week working on it pretty hard" (see the March 2002 AMS Notices).

Introduced 100 years earlier by French mathematician Edouard Lucas and English puzzlist Henry Dudeney, the generalized Tower of Hanoi consists of $n \geq 1$ washer-like pieces with different radii stackable on $k \geq 3$ identical pegs. Beginning with all the pieces on the same peg, the object is to transfer the entire stack to another peg at minimum cost (number of moves) observing these rules:

- a larger piece is never situated on top of a smaller piece;
- only one piece is moved at a time; and
- each piece is on a peg at the end of each move.

First Brain Teaser: Show that the $k = 101$ peg, $n = 5050$ piece game is playable in under 20,000 moves.

Solution: Define a *simple stack of size* j to be j consecutive-sized pieces which are always transferred as a stack from one peg to another in $2j - 1$ consecutive moves. (With at least $j + 1$ available pegs, a simple stack of size j can be transferred by moving its largest piece just once and all others just twice.) We solve the $k = 101$ peg $n = 5050$ piece puzzle using the following simple stacks:

Size of Simple Stack	Number Times Transferred	Number of pieces@Cost per piece
100	2	99@4, 1@2
99	2	98 @4, 1@2
⋮	⋮	⋮
2	2	1@4, 1@2
1	1	1@1

In other words, with 101 available pegs, first transfer the top simple stack with size 100. Then, with 100 available pegs, the next simple stack with size 99, and so on. Move the bottom piece, and then restack. The total cost is $1 + 2(99) + 4(\sum_{i=1}^{99} i) = 19,999$. □

Observe that the i^{th} level (i^{th} largest) piece in a simple stack is always moved with at least $i+1$ available pegs. We may therefore apply to *each* piece in *every* simple stack

The Simple Replacement Rule: For each i, replace the i^{th} level piece p_i in a simple stack by a simple stack with (maximum) size i, whose largest piece has the same size and cost as p_i and whose smaller pieces each cost twice as much as p_i.

For k pegs, applying this rule to all pieces in a single simple stack of size $k-1$ yields $k-1$ new simple stacks with maximum sizes 1 through $k-1$:

level of piece	cost	is replaceable by	max. simple stack size	costs
$k-1$	2	$\rightarrow$	$k-1$	$(k-2)$@4, 1@2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2	2	$\rightarrow$	2	1@4, 1@2
1	1	$\rightarrow$	1	1@1

Playing the multi-peg Tower of Hanoi by transferring simple stacks generated by simple replacement is described in two different ways by Pascal's Triangle situated so each entry sums those in the preceding column at or above the same row:

1	1	1	1
1	2	3	
1	3		
1			

The first way shows the maximum number of pieces at each level generated by simple replacement:

Table 1. Maximum Number of Pieces Generated by Simple Replacement

Level $\downarrow$	Number Pieces $\downarrow$	1	2	$\dots$	j	$j+1$	$\leftarrow$ Iteration of Replacement Rule
$k-1$	1	1	1	$\dots$	1	1	
$k-2$	1	2	3	$\dots$	j	$j+1$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
2	1	$k-2$	$C(k-1,2)$	$\dots$	$C(k-3+j,j)$	$C(k-2+j,j+1)$	$\leftarrow$ Max. Number
1	1	$k-1$	$C(k,2)$	$\dots$	$C(k-2+j,j)$	$C(k-1+j,j+1)$	of Pieces

Observing that the level 2 entry in the $(j+1)^{st}$ column (namely $C(k-2+j,j+1)$) gives the maximum number of new pieces (level 2 or higher) generated by the j^{th} iteration of the replacement rule at a cost of 2^{j+1} per piece, a second Pascal's Triangle describes costs:

Table 2. Costs of Pieces Generated by Simple Replacement

Pegs $\downarrow$	@2^0	@2^1	@2^2	$\dots$	@2^j	@2^{j+1}	$\leftarrow$ Cost per Piece
3	1	1	1	$\dots$	1	1	
4	1	2	3	$\dots$	$C(1+j,j)$	$C(2+j,j+1)$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$k-1$	1	$k-1$	$C(k-2,2)$	$\dots$	$C(k-4+j,j)$	$C(k-3+j,j+1)$	$\leftarrow$ Max. Number
k	1	$k-2$	$C(k-1,2)$	$\dots$	$C(k-3+j,j)$	$C(k-2+j,j+1)$	of Pieces

The first three columns of Table 2 show that the k peg, $n = 1 + (k-2) + C(k-1,2) = k(k-1)/2$ piece game is playable by simple stack transfers at total cost $1 + 2(k-2) + 4C(k-1,2) = 2k^2 - 4k + 1$. This is in fact an optimal solution, since only 1 piece can be moved just once and $k-2$ pieces just twice. The remaining $k(k-1)/2 - (k-1) = C(k-1,2)$ pieces must be move at least four times (twice before and twice after the largest piece is moved). In this manner, we know our answer to the first brain teaser is optimal. We confess (after quite a bit more than a solid week working on it pretty hard!) that we can't answer

Second Brain Teaser: Does Pascal's Triangle always give the optimal total cost?